



# One EURO for Uranus: the Elliptical Uranian Relativity Orbiter mission

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## ABSTRACT

Recent years have seen increasing interest in sending a mission to Uranus, visited so far only by Voyager 2 in 1986. Elliptical Uranian Relativity Orbiter is a preliminary mission concept investigating the possibility of dynamically measuring the planet's angular momentum by means of the Lense–Thirring effect affecting a putative Uranian orbiter. It is possible, at least in principle, to separate the relativistic precessions of the orbital inclination to the celestial equator and of the longitude of the ascending node of the spacecraft from its classical rates of the pericentre induced by the multipoles of the planet's gravity field by adopting an appropriate orbital configuration. For a wide and elliptical  $2000 \times 100\,000$  km orbit, the gravitomagnetic signatures amount to tens of milliarcseconds per year, while, for a suitable choice of the initial conditions, the peak-to-peak amplitude of the range-rate shift can reach the level of  $\simeq 1.5 \times 10^{-3}$  mm s<sup>-1</sup> in a single pericentre passage of a few hours. By lowering the apocentre height to 10 000 km, the Lense–Thirring precessions are enhanced to the level of hundreds of milliarcseconds per year. The uncertainties in the orientation of the planetary spin axis and in the inclination are major sources of systematic bias; it turns out that they should be determined with accuracies as good as  $\simeq 0.1$ –1 and  $\simeq 1$ –10 mas, respectively.

**Key words:** Gravitation – space vehicles – planets and satellites: individual: Uranus.

## 1 INTRODUCTION

So far, Uranus (Bergstrahl, Miner & Matthews 1991; Hubbard 1997), the seventh planet of our Solar system discovered in the 18th century (Herschel 1781), has been visited by an automated spacecraft only once when, in 1986 January, the NASA probe Voyager 2 flew closely past it (Stone & Miner 1986; Stone 1987).

Recent years have seen a renewed interest for the exploration of Uranus and, possibly, Neptune<sup>1</sup> (Cruikshank 1995), discovered in the mid of the 19th century (Galle 1846; Le Verrier 1846), boosting a number of investigations of possible spacecraft-based missions, submitted mostly to NASA and ESA, targeted to at least one or both the two icy giants (National Research Council 2011, 2018, 2022; Mousis et al. 2018; Hofstadter et al. 2019; Fletcher et al. 2020; Gibney 2020; Guillot 2022; Tacconi et al. 2021; Witze 2022). Their main targets are a better knowledge of the fundamental physical parameters of the most distant Sun's planets, such as their gravitational and magnetic fields, rotation rates, and deep atmospheric composition and temperature (Helled, Nettelmann & Guillot 2020), and of their natural satellites, many of them may be icy ocean worlds that could possibly harbour life (Mann 2017). At present, none of such proposals was approved by any space agency.

Here, we make a cursory overview of some of the past proposals. Among them, there was the Uranus Pathfinder concept for a Uranian orbiter (Arridge et al. 2012), submitted in 2011 to ESA as an M-class mission. Origins, Dynamics, and Interiors of the Neptunian and Uranian Systems (ODINUS) was proposed to the ESA's Cosmic Vision programme in 2013 as an L-class mission aimed at sending two twin orbiters around Uranus and Neptune, respectively (Turrini et al. 2014). The science case for an orbital mission to Uranus only, rooted in the ODINUS concept, was explored in a separated study (Arridge et al. 2014). Mission to Uranus for Science and Exploration was another proposal to ESA for an L-class mission to Uranus (Bocanegra-Bahamón et al. 2015). A further mission concept implying a Uranian orbiter, proposed to NASA in 2017, was Origins and Composition of the Exoplanet Analog Uranus System (Mansell et al. 2017). A multiprobe mission to Uranus, including a Small Next-generation Atmospheric Probe, was studied by Sayanagi et al. (2020). Quest to Uranus to Explore Solar system Theories is a lower cost option for a flagship mission aimed to insert an orbiter around Uranus arisen during the 30th Annual NASA/Jet Propulsion Laboratory (JPL) Planetary Science Summer Seminar (Jarmak et al. 2020).

At the time of writing, it seems that the latest proposal for a mission targeted at Uranus, submitted to NASA, is Uranus Orbiter and Probe (Simon, Nimmo & Anderson 2021; Mandt 2023); it is giving rise to a number of related studies (Cohen et al. 2022; Giriya 2023). Preliminary

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<sup>1</sup>In general, missions to Uranus are favoured due to logistical and cost reasons.

hints of a possible flyby of Uranus with a future Chinese interplanetary exploration mission beyond Jupiter can be found in<sup>2</sup> Xu, Zou & Jia (2018).

Among the physical parameters of interest of Uranus (and Neptune), whose determination/constraint is the main goal of the aforementioned proposed missions, there are the (normalized) moment of inertia (MoI)  $\mathcal{J}$  and the rotation period  $P$  (Helled & Fortney 2020; Helled et al. 2020; Neuenschwander & Helled 2022). In general, they depend on the distribution of matter in the interior of a planet, concurring to form its spin angular momentum

$$S = \mathcal{J} M R^2 \omega, \quad (1)$$

where  $M$  is the planet's mass,  $R$  is its equatorial radius, and

$$\omega \doteq \frac{2\pi}{P} \quad (2)$$

is its rotational angular speed. Neuenschwander & Helled (2022) pointed out that a determination of the Uranian MoI  $\mathcal{J}_U$  at a  $\simeq 1$ – $0.1$  per cent level could constrain the planetary rotation period and the depth of the winds, respectively. Neuenschwander & Helled (2022) acknowledged that it is not an easy task which could be implemented only with a future dedicated space mission.

General relativity<sup>3</sup> (Misner, Thorne & Wheeler 2017) offers, in principle, a way to measure the spin angular momentum  $\mathbf{S}$  of a rotating body in a dynamical, model-independent way. Indeed, in its weak-field and slow-motion approximation, its linearized equations formally resemble those of the Maxwellian electromagnetism giving rise to the so-called gravitoelectromagnetic paradigm. For such a concept within the framework of the Einsteinian theory of gravitation, see e.g. Cattaneo (1958), Thorne, MacDonald & Price (1986), Thorne (1986, 1988), Harris (1991), Jantzen, Carini & Bini (1992), Mashhoon (2001, 2007), Rindler (2001), Costa & Herdeiro (2008), Costa & Natário (2014, 2021), Ruggiero (2021), and references therein. Actually, general relativistic gravitoelectromagnetism has nothing to do with electric charges and currents, implying a number of purely gravitational phenomena affecting orbiting test particles, precessing gyroscopes, moving clocks and atoms, and propagating electromagnetic waves (Braginsky, Caves & Thorne 1977; Dymnikova 1986; Ruggiero & Tartaglia 2002; Tartaglia 2002; Schäfer 2004, 2009; Stella & Possenti 2009). In particular, it turns out that matter-energy currents give rise to a ‘gravitomagnetic’ component of the gravitational field encoded in the off-diagonal components  $g_{0i}$ ,  $i = 1, 2, 3$  of the space–time metric tensor  $g_{\mu\nu}$ ,  $\mu, \nu = 0, 1, 2, 3$ . To the first post-Newtonian (1pN) order, in the case of an isolated, slowly spinning body, the source of its gravitomagnetic field is just its spin angular momentum  $\mathbf{S}$  which, among other things, induces a non-central, Lorentz-like acceleration on an orbiting test particle. It causes secular precessions of the orbit of the latter (Soffel 1989; Brumberg 1991; Soffel & Han 2019) which go by the name of Lense–Thirring (LT) effect (Lense & Thirring 1918; Mashhoon, Hehl & Theiss 1984). Gravitomagnetism has been experimentally measured in a, so far, undisputed way only in the field of the spinning Earth with the dedicated GP-B spacecraft-based mission which measured the Pugh–Schiff precessions (Pugh 1959; Schiff 1960) of the axes of four gyroscopes carried onboard to a 19 per cent accuracy (Everitt et al. 2011). As per the LT orbital precessions, somewhat controversial attempts to measure them with the Earth’s artificial satellites of the LAGEOS type (Pearlman et al. 2019) and the satellite laser ranging technique (Coulot et al. 2011) are currently ongoing (Ciufolini et al. 2013; Iorio, Ruggiero & Corda 2013; Renzetti 2013); see Iorio et al. (2011) and references therein also for other proposed tests with natural and artificial bodies in the Solar system. Recently, a successful detection of the gravitomagnetic orbital precessions in a tight astrophysical binary system made of a white dwarf and a pulsar was claimed (Venkatraman Krishnan et al. 2020), but also such a test subsequently raised concerns (Iorio 2020). Efforts for measuring the LT periastron precession of the double pulsar PSR J0737–3039A/B (Burgay et al. 2003; Lyne et al. 2004) in the next future are underway (Kehl et al. 2017; Hu et al. 2020).

Here, we investigate the possibility of measuring the gravitomagnetic LT orbital precessions of a putative Uranian orbiter, provisionally dubbed Elliptical Uranian Relativity Orbiter (EURO), independently of the competing ones due to the even and odd zonal harmonics  $J_\ell$ ,  $\ell = 2, 3, 4, \dots$  of the planet’s gravity field, which are other physical parameters of great interest (Neuenschwander & Helled 2022). The peculiar obliquity of  $98^\circ$  of the Uranian spin axis (Bergstrahl et al. 1991; Hubbard 1997), whose origin is, at present, actively investigated (Rogoszinski & Hamilton 2021; Saillenfest et al. 2022), is instrumental for the achievement of such a goal. For the sake of clarity, the standard Keplerian orbital elements (Murray & Dermott 2000; Bertotti, Farinella & Vokrouhlický 2003a; Kopeikin, Efroimsky & Kaplan 2011) are used to perform a sensitivity analysis, which however is not intended to replace future simulations of the actual observables (such as range-rate), including fitting parameters and a full covariance analysis.

The paper is organized as follows. In Section 2, the LT effect for an arbitrary orientation of the primary’s spin axis is reviewed, while the competing classical precessions induced by the quadrupole mass moment of the central body are treated in Section 3. Section 4 deals with the orbital geometry which better allows separating the aforementioned relativistic and Newtonian effects from each other. In Section 5, numerical values are given for a pair of specific wide and elliptical trajectories of EURO, and the range-rate is considered as well. The impact of the errors in the planetary spin axis orientation and in the orbital inclination on the classical precessions is examined. In Section 6, the conditions and the requirements for the implementation of the previously considered orbits are discussed. Section 7 summarizes our findings and offers our conclusions. Appendix A shows the details of the calculation of the classical orbital precessions due to the first seven zonal harmonics of the

<sup>2</sup>See also <https://www.space.com/china-probes-jupiter-uranus-same-launch> on the Internet.

<sup>3</sup>For a recent overview of the Einstein’s theory of gravity, see e.g. Debono & Smoot (2016) and references therein.

planet's gravity field, while Appendix B is dedicated to the approximate analytical calculation of the range-rate shift of a spacecraft orbiting a distant planet. In Appendix C, some suggestions for properly naming the pericentre and the apocentre in the case of Uranus are offered.

## 2 THE GENERALIZED LT ORBITAL PRECESSIONS

Let an isolated, massive rotating body orbited by a test particle be considered.

The generalized<sup>4</sup> LT averaged rates of the inclination  $I$  of the orbital plane to the reference  $\{x, y\}$  plane adopted, of the longitude of the ascending node  $\Omega$  and of the argument of pericentre  $\omega$  are (Iorio 2017)

$$j^{\text{LT}} = \frac{2GS(\hat{k} \cdot \hat{l})}{c^2 a^3 (1-e^2)^{3/2}}, \quad (3)$$

$$\dot{\Omega}^{\text{LT}} = \frac{2GS \csc I (\hat{k} \cdot \hat{m})}{c^2 a^3 (1-e^2)^{3/2}}, \quad (4)$$

$$\dot{\omega}^{\text{LT}} = -\frac{2GS\hat{k} \cdot (2\hat{h} + \cot I \hat{m})}{c^2 a^3 (1-e^2)^{3/2}}, \quad (5)$$

for other derivations based on different parametrizations (see also Barker & O'Connell 1975; Damour & Schäfer 1988; Damour & Taylor 1992). In equations (3)–(5),  $G$  is the Newtonian constant of gravitation,  $c$  is the speed of light in vacuum,  $\hat{k}$  is the spin axis of the central body so that  $S = S\hat{k}$ ,

$$\hat{h} = \{\sin I \sin \Omega, -\sin I \cos \Omega, \cos I\} \quad (6)$$

is the unit vector of the orbital angular momentum of the test particle (Soffel 1989; Brumberg 1991; Soffel & Han 2019),

$$\hat{l} = \{\cos \Omega, \sin \Omega, 0\} \quad (7)$$

is the unit vector directed along the line of the nodes toward the ascending node (Soffel 1989; Brumberg 1991; Soffel & Han 2019),

$$\hat{m} = \{-\cos I \sin \Omega, \cos I \cos \Omega, \sin I\} \quad (8)$$

is a unit vector lying in the orbital plane (Soffel 1989; Brumberg 1991; Soffel & Han 2019) such that  $\hat{l} \times \hat{m} = \hat{h}$ ,  $a$  is the semimajor axis, and  $e$  is the eccentricity.

Since the reference frame usually adopted for processing Solar system's spacecraft data is the International Celestial Reference Frame (ICRF, Ma et al. 1998; Petit & Luzum 2010), whose reference plane, the celestial equator (CE), coincides with the Earth's equatorial plane at the reference epoch J2000, it is useful to parametrize  $\hat{k}$  in terms of the right ascension (RA)  $\alpha$  and declination (DEC)  $\delta$  of the planet's pole of rotation as

$$\hat{k}_x = \cos \delta \cos \alpha, \quad (9)$$

$$\hat{k}_y = \cos \delta \sin \alpha, \quad (10)$$

$$\hat{k}_z = \sin \delta. \quad (11)$$

By means of equations (9)–(11), equations (3)–(5) can be cast into the form

$$j^{\text{LT}} = \frac{2GS \cos \delta \cos \eta}{c^2 a^3 (1-e^2)^{3/2}}, \quad (12)$$

$$\dot{\Omega}^{\text{LT}} = \frac{2GS(\sin \delta + \cos \delta \cot I \sin \eta)}{c^2 a^3 (1-e^2)^{3/2}}, \quad (13)$$

$$\dot{\omega}^{\text{LT}} = -\frac{2GS[3 \cos I \sin \delta + \cos \delta (\csc I - 3 \sin I) \sin \eta]}{c^2 a^3 (1-e^2)^{3/2}}, \quad (14)$$

where

$$\eta \doteq \alpha - \Omega \quad (15)$$

<sup>4</sup>The formulas obtained originally by Lense & Thirring (1918) and often reported in the literature (Soffel 1989; Brumberg 1991; Soffel & Han 2019) hold when the orientation of  $S$  is known, so that it can be aligned with, say, the reference  $z$ -axis of the coordinate system adopted; in such a case, the reference  $\{x, y\}$  plane coincides with the body's equatorial one.

is the planetary spin's RA reckoned from the line of the nodes. Note that, for

$$\delta = 90^\circ, \quad (16)$$

the usual formulas by Lense & Thirring (1918),

$$i^{\text{LT}} = 0, \quad (17)$$

$$\dot{\Omega}^{\text{LT}} = \frac{2GS}{c^2 a^3 (1-e^2)^{3/2}}, \quad (18)$$

$$\dot{\omega}^{\text{LT}} = -\frac{6GS \cos I}{c^2 a^3 (1-e^2)^{3/2}}, \quad (19)$$

valid, e.g. for an Earth's satellite, are restored.

In the case of Uranus, the general formulas of equations (3)–(5), or equations (12)–(14), are required since the orientation of its spin axis with respect to the ICRF is given by (Jacobson 2014)

$$\alpha_{\delta} = 77.310^\circ, \quad (20)$$

$$\delta_{\delta} = 15.172^\circ. \quad (21)$$

Thus,

$$\hat{k}_x^{\delta} = 0.212, \quad (22)$$

$$\hat{k}_y^{\delta} = 0.941, \quad (23)$$

$$\hat{k}_z^{\delta} = 0.262; \quad (24)$$

the spin of Uranus mainly lies in the CE. As consequences, equations (17)–(19) cannot be used in this case.

### 3 THE ORBITAL PRECESSIONS INDUCED BY $J_2$

The first even zonal harmonic coefficient  $J_2$  of the multipolar expansion of the classical part of the non-spherical planetary gravitational potential  $U(\mathbf{r})$  induces the largest orbital precessions among those due to the deviations of  $U$  from spherical symmetry (Capderou 2005); they act as major systematic bias with respect to the relativistic rates of equations (3)–(5).

For an arbitrary<sup>5</sup> orientation of  $\hat{\mathbf{k}}$  in space, the oblateness-driven averaged rates are (Iorio 2017)

$$i^{J_2} = -\frac{3}{2} n_b J_2 \left(\frac{R}{p}\right)^2 (\hat{\mathbf{k}} \cdot \hat{\mathbf{l}}) (\hat{\mathbf{k}} \cdot \hat{\mathbf{h}}), \quad (25)$$

$$\dot{\Omega}^{J_2} = -\frac{3}{2} n_b J_2 \left(\frac{R}{p}\right)^2 \csc I (\hat{\mathbf{k}} \cdot \hat{\mathbf{m}}) (\hat{\mathbf{k}} \cdot \hat{\mathbf{h}}), \quad (26)$$

$$\dot{\omega}^{J_2} = \frac{3}{4} n_b J_2 \left(\frac{R}{p}\right)^2 \left\{ 2 - 3 \left[ (\hat{\mathbf{k}} \cdot \hat{\mathbf{l}})^2 + (\hat{\mathbf{k}} \cdot \hat{\mathbf{m}})^2 \right] + 2 \cot I (\hat{\mathbf{k}} \cdot \hat{\mathbf{m}}) (\hat{\mathbf{k}} \cdot \hat{\mathbf{h}}) \right\}, \quad (27)$$

where

$$n_b \doteq \sqrt{\frac{GM}{a^3}} \quad (28)$$

is the Keplerian mean motion, and

$$p \doteq a (1 - e^2) \quad (29)$$

is the orbital semilatus rectum.

<sup>5</sup>The same considerations as in footnote 4 hold here for the oblateness-induced orbital precessions. Their expressions for  $\hat{\mathbf{k}}$  aligned with the reference  $z$ -axis can be found, e.g. in Capderou (2005); they are reported in equations (34)–(36).

By means of equations (9)–(11), equations (25)–(27) can be suitably expressed as

$$J^{J_2} = \frac{3}{2} n_b J_2 \left(\frac{R}{p}\right)^2 [\cos \delta \cos \eta (-\cos I \sin \delta + \cos \delta \sin I \sin \eta)], \quad (30)$$

$$\dot{\Omega}^{J_2} = -\frac{3}{2} n_b J_2 \left(\frac{R}{p}\right)^2 \csc I (\sin I \sin \delta + \cos I \cos \delta \sin \eta) (\cos I \sin \delta - \cos \delta \sin I \sin \eta), \quad (31)$$

$$\begin{aligned} \dot{\omega}^{J_2} = -\frac{3}{8} n_b J_2 \left(\frac{R}{p}\right)^2 [-4 + (1 - 5 \cos 2I) \sin^2 \delta + (3 - 5 \cos 2I) \cot I \sin 2\delta \sin \eta + \\ + 2 \cos^2 \delta (3 \cos^2 \eta + 5 \cos^2 I \sin^2 \eta)]. \end{aligned} \quad (32)$$

Note that, for

$$\delta = 90^\circ, \quad (33)$$

equations (30)–(32) reduce to the standard formulas (Capderou 2005)

$$J^{J_2} = 0, \quad (34)$$

$$\dot{\Omega}^{J_2} = -\frac{3}{2} n_b J_2 \left(\frac{R}{p}\right)^2 \cos I, \quad (35)$$

$$\dot{\omega}^{J_2} = \frac{3}{8} n_b J_2 \left(\frac{R}{p}\right)^2 (3 + 5 \cos 2I). \quad (36)$$

valid, e.g. for an Earth's satellite.

The details of the calculation of the precessions induced by the zonal harmonics  $J_\ell$  of any degree  $\ell$  can be found in Appendix A.

#### 4 CHOOSING THE MOST SUITABLE ORBITAL GEOMETRY

From equations (3)–(5) and equations (25)–(27), it turns out that, if

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{h}} = 0, \quad (37)$$

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{m}} \neq 0, \quad (38)$$

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{i}} \neq 0, \quad (39)$$

i.e. if the body's spin axis lies in the orbital plane of the test particle somewhere between the line of the nodes and the direction perpendicular to the latter, the  $J_2$ -induced shifts of  $I$  (equation 25) and  $\Omega$  (equation 26) vanish, contrary to the corresponding gravitomagnetic ones of equations (3)–(4). Then, the relativistic node and inclination precessions would not be biased at all by the classical disturbances due to the body's oblateness. The condition of equation (37), corresponding to a polar orbit, is usually selected in order to fulfil some of the planetological scopes of most of the proposed Uranian missions (see e.g. Simon et al. 2021; Girija 2023).

By means of equations (6)–(11), the conditions of equations (37)–(39) are equivalent to

$$\tan I = \csc \eta \tan \delta, \quad (40)$$

$$\tan I \neq -\sin \eta \cot \delta, \quad (41)$$

$$\cos \eta \cos \delta \neq 0. \quad (42)$$

In the case of Uranus, according to equation (21), it is

$$\sin \delta_\mathcal{U} = 0.262 \neq 0, \quad (43)$$

$$\cos \delta_{\delta} = 0.965 \neq 0. \quad (44)$$

Thus, for EURO, equations (40)–(42) can be simultaneously fulfilled if the planetary spin’s RA relative to the node

$$\eta \neq 90^{\circ}. \quad (45)$$

In particular, if one chooses

$$\eta = 0, \quad (46)$$

then, from equations (40)–(42), it turns out that

$$I = 90^{\circ}, \quad (47)$$

implying that the gravitomagnetic pericentre precession of equation (14) vanishes, contrary to the classical one of equation (32). Indeed, equations (12)–(13) and equation (32), calculated with equations (46)–(47), reduce to the following secular rates:

$$\dot{I}^{\text{LT}} = \frac{2GS \cos \delta}{c^2 a^3 (1-e^2)^{3/2}}, \quad (48)$$

$$\dot{\Omega}^{\text{LT}} = \frac{2GS \sin \delta}{c^2 a^3 (1-e^2)^{3/2}}, \quad (49)$$

$$\dot{\omega}^{J_2} = -\frac{3}{4} n_b J_2 \left( \frac{R}{p} \right)^2. \quad (50)$$

Furthermore, as shown in Appendix A, the averaged classical rates of  $I$  and  $\Omega$  vanish also for the even and odd zonals of degree  $\ell > 2$ , contrary to the pericentre  $\omega$  which turns out to be impacted by the odd zonals as well. Thus, in principle, one could separate the general relativistic signatures of  $I$  and  $\Omega$  from the Newtonian ones of due to  $J_{\ell}$ ,  $\ell = 2, 3, 4, \dots$  which could be used to simultaneously determine the multipolar field of Uranus without any a priori ‘imprinting’ due to its 1pN gravitomagnetic field, and vice versa:  $S$  and  $J_{\ell}$ ,  $\ell = 2, 3, 4, \dots$  could, then, be measured independently of each other. It would be important since, although there is no strict one-to-one correspondence between  $J_2^{\delta}$  and the Uranus’ MoI (see section 3.3 of Neuenschwander & Helled 2022), there is a rather strong correlation between the other Uranian multipolar coefficients  $J_4$ ,  $J_6$ ,  $J_8$ , and the MoI (Neuenschwander & Helled 2022). General relativity would, instead, imprint the classical pericentre precessions through the mismodelled part of its ‘gravitoelectric’ 1pN counterpart given by (Soffel 1989; Brumberg 1991; Soffel & Han 2019)

$$\omega^{\text{GE}} = \left( \frac{2 + 2\gamma - \beta}{3} \right) \frac{3 n_b \mu}{c^2 p}, \quad (51)$$

where  $\gamma$ ,  $\beta$  are the parametrized post-Newtonian (PPN) parameters (Will & Nordtvedt 1972; Will 2018) which are equal to unity in general relativity.

## 5 THE CASE OF A POLAR, HIGHLY ECCENTRIC URANIAN ORBITER

By calculating equation (1) with (table 1 of Neuenschwander & Helled 2022)

$$P_{\delta} = 17.24 \text{ h} \quad (52)$$

and (table 2 of Neuenschwander & Helled 2022)

$$\mathcal{J}_{\delta} = 0.22594, \quad (53)$$

along with the reference radius for the Uranian zonal harmonics (French et al. 1988)

$$R_{\delta} = 25\,559 \text{ km} \quad (54)$$

and<sup>6</sup> (Jacobson 2014)

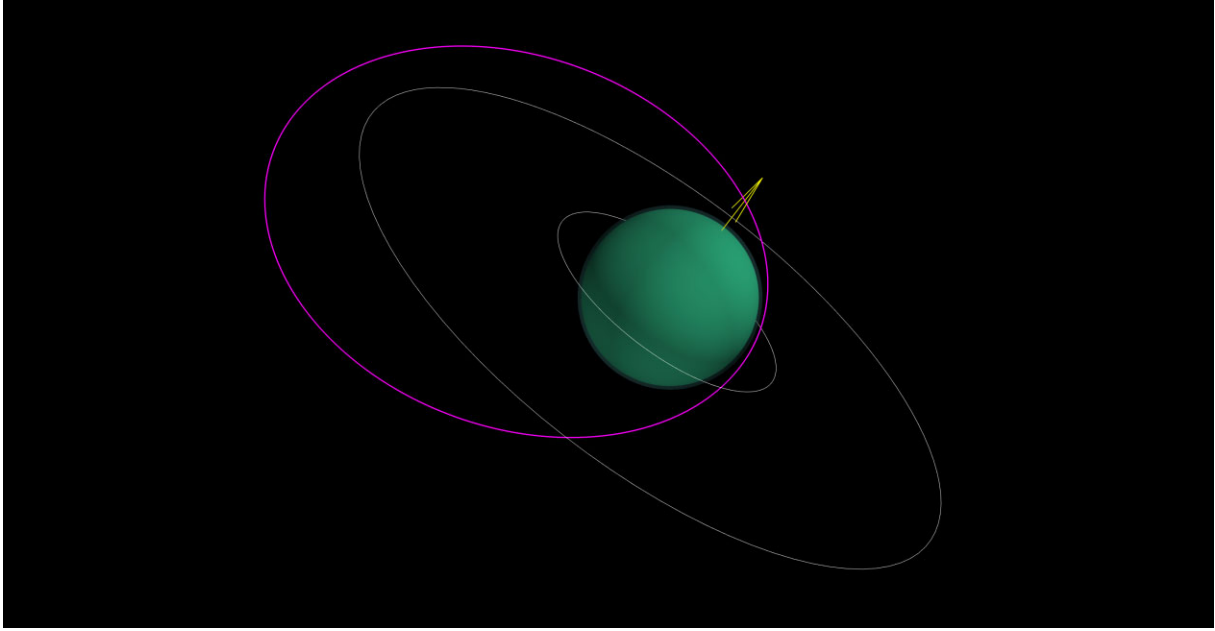
$$\mu_{\delta} = 5794\,556.4 \times 10^9 \text{ m}^3 \text{ s}^{-2}, \quad (55)$$

a reasonable guess for the Uranian spin angular momentum may be

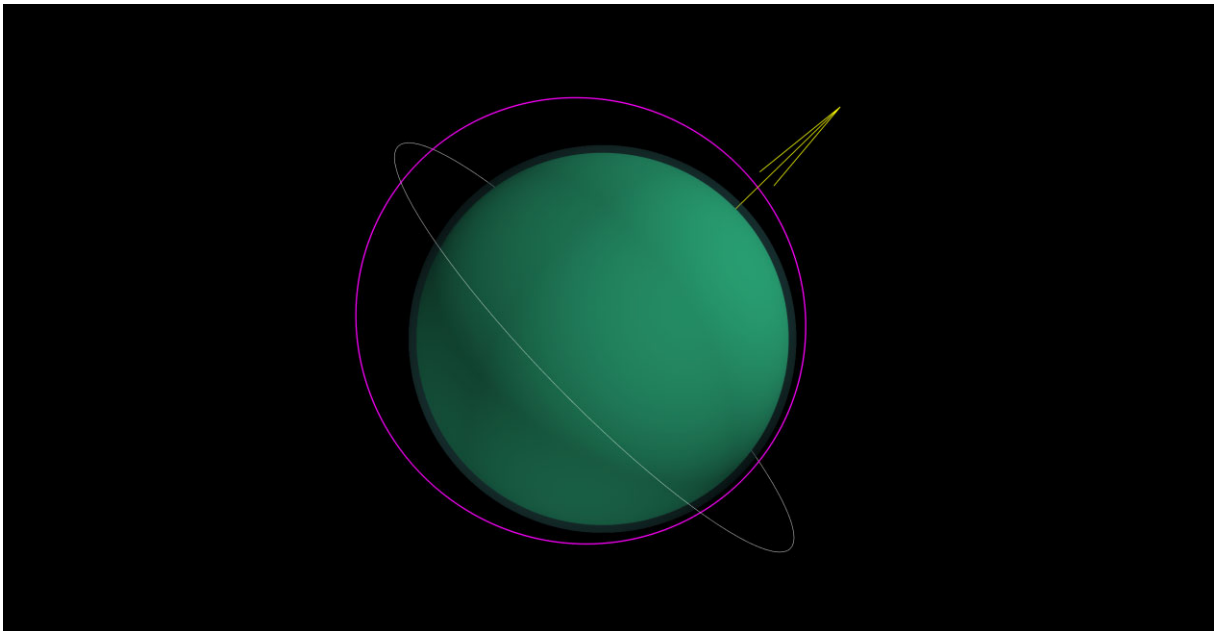
$$S_{\delta} \simeq 1.29 \times 10^{36} \text{ kg m}^2 \text{ s}^{-1}. \quad (56)$$

It turns out that calculating equation (1) with the other values for  $P_{\delta}$  and  $\mathcal{J}_{\delta}$  in Neuenschwander & Helled (2022, tables 1–2) yields a  $\simeq 1$  per cent uncertainty in the theoretically predicted value of  $S_{\delta}$ .

<sup>6</sup>The values of equations (54)–(55) are reported also in Neuenschwander & Helled (2022, table 1).



**Figure 1.** Orbital geometry of EURO for a  $2000 \times 100\,000$  km orbit (purple) oriented with respect to the ICRF according to equations (46)–(47). The Uranian spin axis, in yellow, is oriented as per equations (20)–(21) with respect to the CE. The white circles represent the innermost (1986U2R/ $\zeta$ ) and outermost ( $\mu$ ) rings of Uranus, respectively.



**Figure 2.** Orbital geometry of EURO for a  $2000 \times 10\,000$  km orbit oriented with respect to the ICRF according to equations (46)–(47). The Uranian spin axis, in yellow, is oriented as per equations (20)–(21) with respect to the CE. The white circles represent the innermost (1986U2R/ $\zeta$ ) and outermost ( $\mu$ ) rings of Uranus, respectively.

As far as the shape and the size of the spacecraft’s path are concerned, wide and highly elliptical orbits are usually adopted for the orbital phase of several proposed missions. Just as an example for Uranus, a  $4000 \times 550\,000$  km orbit was recently proposed by Girija (2023) in several simulations.

Here, a lower and smaller  $2000 \times 100\,000$  km path, corresponding to

$$a = 76\,559 \text{ km}, e = 0.6400, P_b = 15.3 \text{ h}, \tag{57}$$

where  $P_b$  is the orbital period of the probe, will be examined. Fig. 1 depicts such kind of orbit (purple) oriented according to equations (46)–(47).

A further scenario, shown in Fig. 2, encompasses a less eccentric orbit with apocentre height  $h_{apo} = 10\,000$  km, corresponding

to

$$a = 31\,559 \text{ km}, \quad e = 0.1267, \quad P_b = 4.06 \text{ h}. \quad (58)$$

For details on the practical implementation of such orbital geometries, see Section 6.

### 5.1 The orbital precessions

By assuming (Jacobson 2014)

$$J_2^{\hat{\delta}} (\times 10^6) = 3510.7, \quad (59)$$

equations (48)–(50) and<sup>7</sup> equation (51), calculated with the orbital parameters of equation (57), yield

$$i^{\text{LT}} \simeq 59.5 \text{ mas yr}^{-1}, \quad (60)$$

$$\dot{\Omega}^{\text{LT}} \simeq 16.1 \text{ mas yr}^{-1}, \quad (61)$$

$$\dot{\omega}^{J_2} = -173.0^\circ \text{ yr}^{-1}, \quad (62)$$

$$\dot{\omega}^{\text{GE}} = 3.2 \text{ arcsec yr}^{-1}, \quad (63)$$

where  $\text{mas yr}^{-1}$  and  $\text{arcsec yr}^{-1}$  stand for milliarcseconds per year and arcseconds per year, respectively.

The  $2000 \times 10\,000 \text{ km}$  orbit of equation (58) yields

$$i^{\text{LT}} \simeq 394.6 \text{ mas yr}^{-1}, \quad (64)$$

$$\dot{\Omega}^{\text{LT}} \simeq 107.0 \text{ mas yr}^{-1}, \quad (65)$$

$$\dot{\omega}^{J_2} = -1\,384.9^\circ \text{ yr}^{-1}, \quad \text{and} \quad (66)$$

$$\dot{\omega}^{\text{GE}} = 17.4 \text{ arcsec yr}^{-1}. \quad (67)$$

Fig. 3 displays the nominal orbital precessions as per equations (48)–(51) along with the semimajor axis  $a$ , in km, and the eccentricity  $e$  as functions of  $h_{\text{apo}}$ , ranging from 2000 to 100 000 km, for a fixed value of the pericentre height  $h_{\text{peri}} = 2000 \text{ km}$ .

About the ‘imprinting’ of the mismodelling in the 1pN gravitoelectric pericentre precession on the classical one, from equation (51) it turns out that, in principle, it is mainly due to the errors in the PPN parameters  $\gamma$  and  $\beta$ , currently known at the  $\simeq 10^{-5}$  accuracy level (Bertotti, Iess & Tortora 2003b; Pitjeva & Pitjev 2014; Fienga et al. 2015; Park et al. 2017; Genova et al. 2018). Instead, the error in the Uranian gravitational parameter  $\mu_{\hat{\delta}}$  would be of less concern. Indeed, according to Jacobson (2014), its current relative uncertainty amounts to just

$$\frac{\sigma_{\mu_{\hat{\delta}}}}{\mu_{\hat{\delta}}} = 7 \times 10^{-7}. \quad (68)$$

However, equations (62)–(63) and the pictures in the middle row of Fig. 3 show that the issue of the 1pN bias on the determination of  $J_2^{\hat{\delta}}$  is negligible; suffice it to say that the current relative uncertainty in  $J_2^{\hat{\delta}}$  is  $2 \times 10^{-4}$ , as per table 12 of Jacobson (2014).

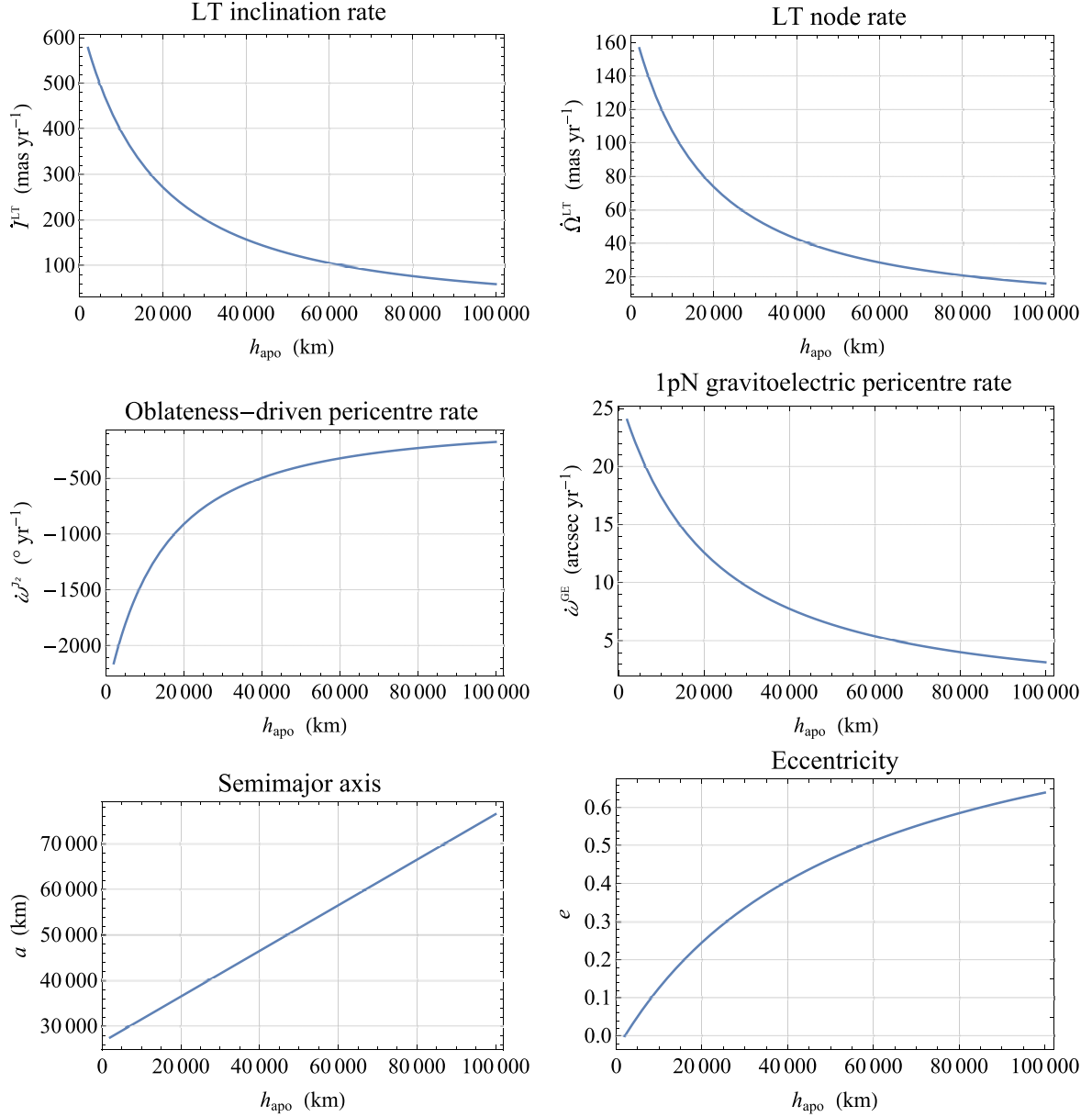
#### 5.1.1 The impact of the uncertainty in the pole of Uranus

Here, the consequences of the current level of uncertainty in our knowledge of the orientation of the Uranian spin axis on the otherwise idealized situation outlined in Section 4 is examined.

At present, the pole of Uranus is known to an accuracy of (Jacobson 2014)

<sup>7</sup>The gravitoelectric pericentre precession was computed for  $\gamma = \beta = 1$ .





**Figure 3.** Upper row: LT precessions of the inclination  $I$  (left-hand panel) and of the node  $\Omega$  (right-hand panel), in milliarcseconds per year, calculated from equations (48)–(49) as functions of  $h_{\text{apo}}$ , in km, for a fixed value of  $h_{\text{peri}} = 2000$  km. For  $\delta_{\delta}$  and  $S_{\delta}$ , equation (21) and equation (56) were adopted, respectively. Middle row: Oblateness-driven (left-hand panel) and 1pN gravitoelectric (right-hand panel) precessions of the pericentre  $\omega$ , in degrees per year and arcseconds per year, respectively, calculated from equations (50)–(51) as functions of  $h_{\text{apo}}$ , in km, for a fixed value of  $h_{\text{peri}} = 2000$  km. For  $R_{\delta}$  and  $J_2^{\delta}$ , equation (54) and equation (59) were adopted, respectively. Lower row: Semimajor axis  $a$ , in km, (left-hand panel) and eccentricity  $e$  (right-hand panel) calculated for as functions of  $h_{\text{apo}}$ , in km, for a fixed value of  $h_{\text{peri}} = 2000$  km.

$$\sigma_{\alpha_{\delta}} = 0.002^{\circ} = 7.2 \text{ arcsec}, \quad (69)$$

$$\sigma_{\delta_{\delta}} = 0.002^{\circ} = 7.2 \text{ arcsec}. \quad (70)$$

Such a somewhat modest accuracy is due to the present lack of dedicated, *in situ* missions, contrary to the case, of, e.g. Cassini at Saturn (Spilker 2019); for the Kronian spin axis, Jacobson (2022) recently obtained

$$\sigma_{\alpha_Y} \simeq 0.0001^{\circ} = 0.36 \text{ arcsec}, \quad (71)$$

$$\sigma_{\delta_Y} \simeq 0.00003^{\circ} = 0.1 \text{ arcsec}. \quad (72)$$

The mismodelled quadrupole-induced orbital precessions due to the uncertainty in the primary's spin axis can be analytically evaluated as

$$\sigma_{\kappa J_2} = \sqrt{\left(\frac{\partial \kappa J_2}{\partial \alpha}\right)^2 \sigma_\alpha^2 + \left(\frac{\partial \kappa J_2}{\partial \delta}\right)^2 \sigma_\delta^2}, \quad \kappa = I, \Omega, \quad (73)$$

starting from equations (30)–(31).

For the  $2000 \times 100\,000$  km orbit of equation (57), equation (73), applied to equations (30)–(31) with equations (46)–(47) and equations (69)–(70), yields

$$\sigma_{I J_2} = 40\,502 \text{ mas yr}^{-1}, \quad (74)$$

$$\sigma_{\Omega J_2} = 10\,983 \text{ mas yr}^{-1}, \quad (75)$$

which are about 3 orders of magnitude larger than the nominal LT rates of equations (60)–(61).

In the case of the less eccentric  $2000 \times 10\,000$  km orbit of equation (58), one has

$$\sigma_{I J_2} = 324\,220 \text{ mas yr}^{-1}, \quad (76)$$

$$\sigma_{\Omega J_2} = 87\,918 \text{ mas yr}^{-1}, \quad (77)$$

which are  $\simeq 2$ – $3$  orders of magnitude larger than equations (64)–(65).

Thus,  $\alpha_\delta \delta_\delta$  should be determined with an error as little as

$$\sigma_{\alpha_\delta} \simeq \sigma_{\delta_\delta} \simeq 0.1\text{--}1 \text{ mas}. \quad (78)$$

A further potential challenge is represented by the uncertainty  $\sigma_I$  in the orbital inclination  $I$  to the CE; applying equation (73) to such a potentially major source of systematic bias yields that  $I$  would need to be determined with an accuracy as good as

$$\sigma_I \simeq 1\text{--}10 \text{ mas}. \quad (79)$$

## 5.2 The range-rate and rate shifts

One of the most accurately measured observable in a spacecraft-based mission to a planet P is the Earth-probe range-rate  $\dot{\rho}$ . As an example, the two-way  $Ka$ -band Doppler range-rate measurements of the spacecraft Juno (Bolton 2018), currently orbiting Jupiter, are accurate to a  $\simeq 1 \times 10^{-2} \text{ mm s}^{-1}$  level over 60 s after having processed data covering up to the middle of its prime mission (Durante et al. 2020).

For the computational strategy to recover the shift  $\Delta \dot{\rho}$  of the dynamical part of the range-rate due to the planetocentric orbital motion affected by a perturbing acceleration, see Appendix B.

In the case of the LT effect, the gravitomagnetic acceleration  $A_{\text{LT}}$  can be found, e.g. in Soffel (1989), Petit & Luzum (2010), and Soffel & Han (2019), and its radial, transverse, and out-of-plane components in terms of the unit vectors  $\hat{l}$ ,  $\hat{m}$ , and  $\hat{h}$  are explicitly displayed in Iorio (2017). The resulting relevant gravitomagnetic shifts, calculated for the particular orbital geometry of equations (46)–(47), turn out to be

$$\Delta a^{\text{LT}}(f) = 0, \quad (80)$$

$$\Delta e^{\text{LT}}(f) = 0, \quad (81)$$

$$\begin{aligned} \Delta I^{\text{LT}}(f) = & \frac{GS}{2c^2 n_b a^3 (1-e^2)^{3/2}} \{4(f-f_0) \cos \delta + 4 \cos(f+f_0-\delta+2\omega) \sin(f-f_0) + \\ & + e [3 \sin(f-\delta) - 3 \sin(f_0-\delta) + \sin(f+\delta) - \sin(f_0+\delta) + \sin(f-\delta+2\omega) + \\ & + \sin(3f-\delta+2\omega) - 2 \cos f_0 \sin(2f_0-\delta+2\omega)]\}, \end{aligned} \quad (82)$$

$$\begin{aligned} \Delta \Omega^{\text{LT}}(f) = & -\frac{GS}{2c^2 n_b a^3 (1-e^2)^{3/2}} \{2 \cos(2f-\delta+2\omega) - 2 \cos(2f_0-\delta+2\omega) + \\ & + e [-3 \cos(f-\delta) + 3 \cos(f_0-\delta) + \cos(f+\delta) - \cos(f_0+\delta) + \cos(f-\delta+2\omega) + \\ & + \cos(3f-\delta+2\omega) - 2 \cos f_0 \cos(2f_0-\delta+2\omega)] + 4(-f+f_0) \sin \delta\}, \end{aligned} \quad (83)$$

$$\Delta \omega^{\text{LT}}(f) = 0, \quad (84)$$

$$\Delta \mathcal{M}^{\text{LT}}(f) = 0, \quad (85)$$

where  $f$  is the true anomaly reckoning the instantaneous position of the test particle along its orbit,  $f_0$  is the value of the true anomaly at some initial instant  $t_0$ , and  $\mathcal{M}$  is the mean anomaly. Thus, for the LT perturbations of the radial ( $R$ ), transverse ( $T$ ), and out-of-plane ( $N$ ) components of the velocity one has

$$\Delta v_{\text{R}}^{\text{LT}}(f) = 0, \quad (86)$$

$$\Delta v_{\text{T}}^{\text{LT}}(f) = 0, \quad (87)$$

$$\begin{aligned} \Delta v_{\text{N}}^{\text{LT}}(f) = & \frac{GS}{2c^2 a^2 (1-e^2)^2} (4e(f-f_0) \cos(\delta-\omega) + 4(f-f_0) \cos(f-\delta+\omega) + \\ & + 4 \cos(f_0-\delta+\omega) \sin(f-f_0) + e \{ \sin(f-3f_0+\delta-\omega) + \sin(f-f_0+\delta-\omega) + \\ & + 6 \sin(2f-\delta+\omega) + \sin(f-f_0-\delta+\omega) - 3 \sin(f+f_0-\delta+\omega) - \\ & - 2 \sin(2f_0-\delta+\omega) + e [\sin(f+\delta-\omega) - \sin(f_0+\delta-\omega) + \\ & + 4 \sin(f-\delta+\omega) + \sin(3f-\delta+\omega) - 4 \sin(f_0-\delta+\omega) - \sin(3f_0-\delta+\omega)]). \end{aligned} \quad (88)$$

Inserting equations (86)–(88) in equation (B23) allows to obtain the instantaneous LT range-rate shift;

$$\begin{aligned} \Delta \dot{\rho}^{\text{LT}}(f) = & -\frac{GS \cos \chi \sin(\alpha-\phi)}{2c^2 a^2 (1-e^2)^2} (4e(f-f_0) \cos(\delta-\omega) + 4(f-f_0) \cos(f-\delta+\omega) + \\ & + 4 \cos(f_0-\delta+\omega) \sin(f-f_0) + e \{ \sin(f-3f_0+\delta-\omega) + \sin(f-f_0+\delta-\omega) + \\ & + 6 \sin(2f-\delta+\omega) + \sin(f-f_0-\delta+\omega) - 3 \sin(f+f_0-\delta+\omega) - \\ & - 2 \sin(2f_0-\delta+\omega) + e [\sin(f+\delta-\omega) - \sin(f_0+\delta-\omega) + \\ & + 4 \sin(f-\delta+\omega) + \sin(3f-\delta+\omega) - 4 \sin(f_0-\delta+\omega) - \sin(3f_0-\delta+\omega)]), \end{aligned} \quad (89)$$

where  $\phi$  and  $\chi$  are the RA and DEC of the planet, respectively.

After having averaged equation (89) over one orbital revolution as

$$\langle \Delta \dot{\rho}^{\text{LT}} \rangle = \frac{n_{\text{b}}}{2\pi} \int_{f_0}^{f_0+2\pi} \Delta \dot{\rho}^{\text{LT}}(f) \frac{dt}{df} df, \quad (90)$$

and considering equation (A6), one finally gets for the net gravitomagnetic range-rate shift per orbit

$$\langle \Delta \dot{\rho}^{\text{LT}} \rangle = -\frac{2GS \cos \chi \sin(\alpha-\phi) \sin(u_0-\delta)}{c^2 a^2 \sqrt{1-e^2} (1+e \cos f_0)}. \quad (91)$$

The RA  $\phi_{\delta}$  and DEC  $\chi_{\delta}$  of Uranus, retrieved through the web interface HORIZONS maintained by the NASA JPL for the epoch, say, 2032 December 13, are

$$\phi_{\delta} = 89.455^\circ, \quad (92)$$

$$\chi_{\delta} = 23.6497^\circ. \quad (93)$$

equation (91), calculated with equation (20), equations (56)–(57), and equations (92)–(93), yields

$$\langle \Delta \dot{\rho}^{\text{LT}} \rangle \left[ \frac{1+e \cos f_0}{\sin(u_0-\delta)} \right] = -\frac{2GS_{\delta} \cos \chi_{\delta} \sin(\alpha_{\delta}-\phi_{\delta})}{c^2 a^2 \sqrt{1-e^2}} = 8 \times 10^{-5} \text{ mm s}^{-1}. \quad (94)$$

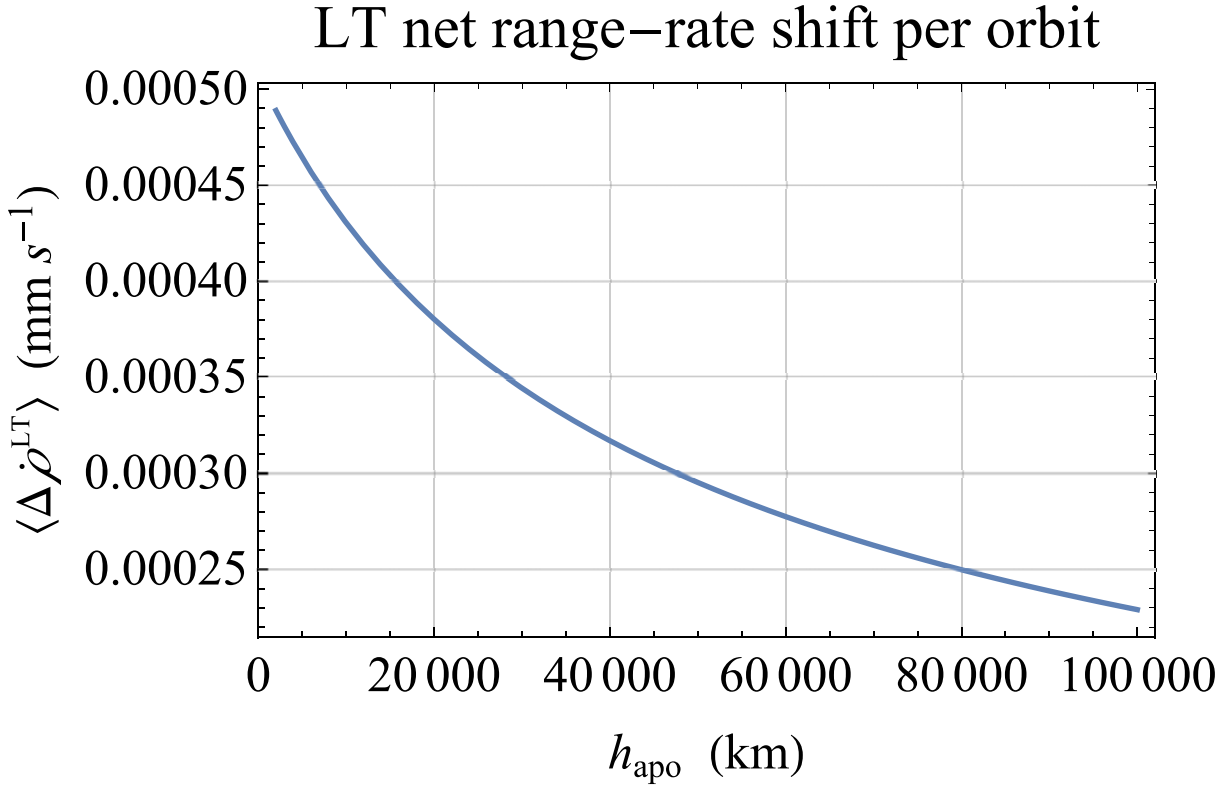
By selecting

$$f_0 = 180^\circ, \quad (95)$$

$$\omega = \delta_{\delta} - 90^\circ, \quad (96)$$

it is possible to maximize equation (94), thus getting

$$\langle \Delta \dot{\rho}^{\text{LT}} \rangle = -\frac{2GS_{\delta} \cos \chi_{\delta} \sin(\alpha_{\delta}-\phi_{\delta})}{c^2 a^2 \sqrt{1-e^2} (1-e)} = 2 \times 10^{-4} \text{ mm s}^{-1}. \quad (97)$$



**Figure 4.** LT net shift per orbit  $\langle \Delta \dot{\rho}^{\text{LT}} \rangle$ , in millimetre per second, of the range-rate of EURO calculated from equation (97) as a function of  $h_{\text{apo}}$ , in km, for a fixed value of  $h_{\text{peri}} = 2000$  km. For  $\alpha_{\delta}$ ,  $S_{\delta}$ ,  $\phi_{\delta}$ ,  $\chi_{\delta}$ , equation (20), equation (56), and equations (92)–(93) were adopted, respectively.

Fig. 4 plots equation (97), in millimetre per second, as a function of the apocentre height  $h_{\text{apo}}$ , ranging from 2000 to 100 000 km, for a fixed value of the pericentre height  $h_{\text{peri}} = 2000$  km.

It turns out that an apocentre height  $h_{\text{apo}} \simeq 50\,000$  km allows reaching the  $\simeq 3 \times 10^{-4}$  mm s<sup>-1</sup> level; for  $h_{\text{apo}} = 10\,000$  km, the LT net range-rate shift is of the order of  $\simeq 4 \times 10^{-4}$  mm s<sup>-1</sup>.

Fig. 5 depicts the instantaneous LT range-rate shift  $\Delta \dot{\rho}^{\text{LT}}(t)$  during a single passage at the pericentre 6 h long for the orbital configuration of equation (57) and equations (95)–(96). To this end, we expressed  $\Delta \dot{\rho}^{\text{LT}}(f)$  from equation (89) as a function of time by switching from  $f$  to time  $t$  through  $\mathcal{M}$  according to Brouwer & Clemence (1961):

$$f(t) = \mathcal{M}(t) + 2 \sum_{s=1}^{s_{\text{max}}} \frac{1}{s} \left\{ J_s(s e) + \sum_{j=1}^{j_{\text{max}}} \lambda^j [J_{s-j}(s e) + J_{s+j}(s e)] \right\} \sin(s \mathcal{M}(t)), \quad (98)$$

where

$$\mathcal{M}(t) = \mathcal{M}_0 + n_b (t - t_0), \quad (99)$$

$$\lambda = \frac{1 - \sqrt{1 - e^2}}{e}, \quad (100)$$

$\mathcal{M}_0$  is the mean anomaly at the initial epoch  $t_0$ , and  $J_k(s e)$  is the Bessel function of the first kind of order  $k$ . The initial instant  $t_0$  corresponds to the passage at apocentre, as per equation (95), so that

$$\mathcal{M}_0 = 180^\circ; \quad (101)$$

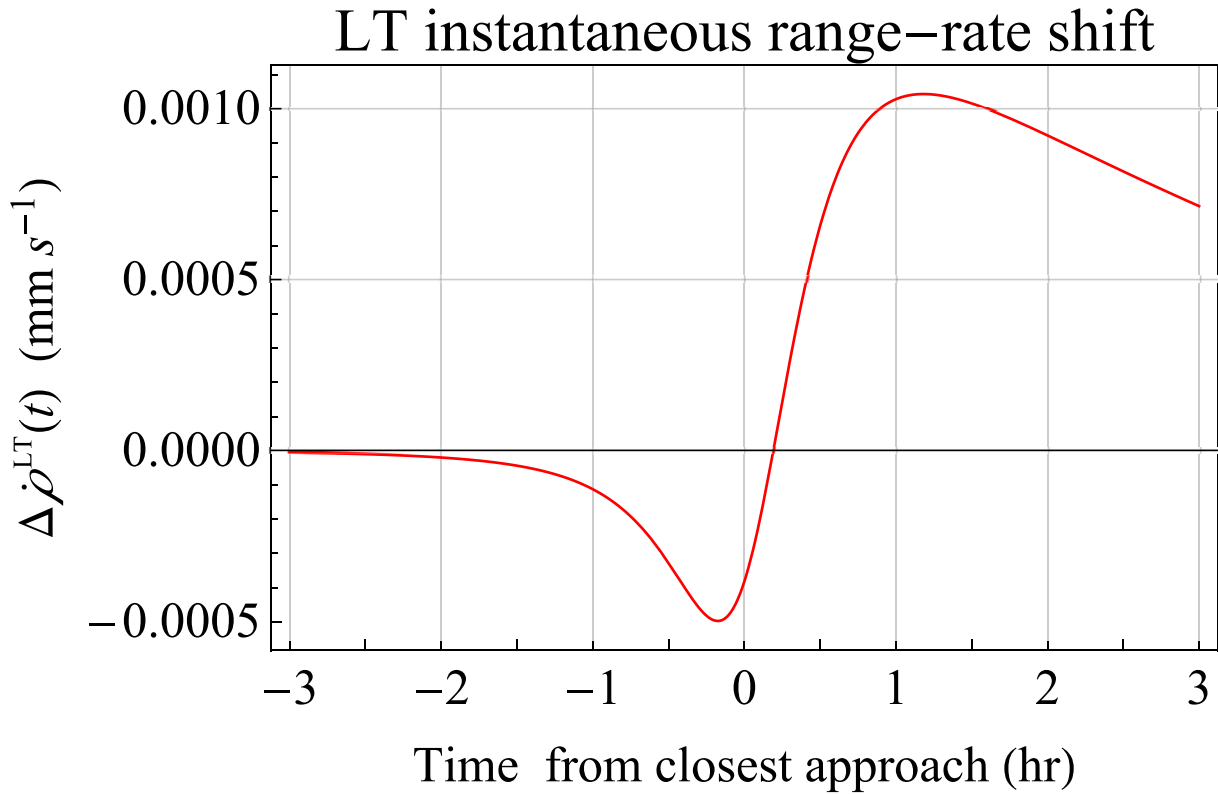
the pericentre is reached after half an orbital period.

It can be noted that the peak-to-peak amplitude of the signal amounts to  $\simeq 1.5 \times 10^{-3}$  mm s<sup>-1</sup>.

Such figures are challenging if compared with the  $\simeq 10^{-2}$  mm s<sup>-1</sup> accuracy level of Juno.

## 6 ORBITAL CONFIGURATIONS AND REQUIREMENTS

Elliptical orbits are generally preferred in mission design, since they allow reducing greatly the amount of propellant needed to impart the large change  $\Delta V$  in speed required to finally insert the spacecraft in an circular, low trajectory (Hughes 2016). Moreover, it is also mandatory



**Figure 5.** LT instantaneous shift  $\Delta\rho^{\text{LT}}(t)$ , in millimetre per second, of the range-rate of EURO calculated from equation (89) with equation (57), corresponding to a  $2000 \times 100\,000$  km orbit, and equations (95)–(96) as a function of time  $t$ , according to equation (98) for  $s_{\text{max}} = j_{\text{max}} = 55$ , during a passage 6 h long centred at the pericentre. For  $\alpha_{\delta}$ ,  $\delta_{\delta}$ ,  $S_{\delta}$ ,  $\phi_{\delta}$ ,  $\chi_{\delta}$ , equations (20)–(21), equation (56), and equations (92)–(93), were adopted, respectively.

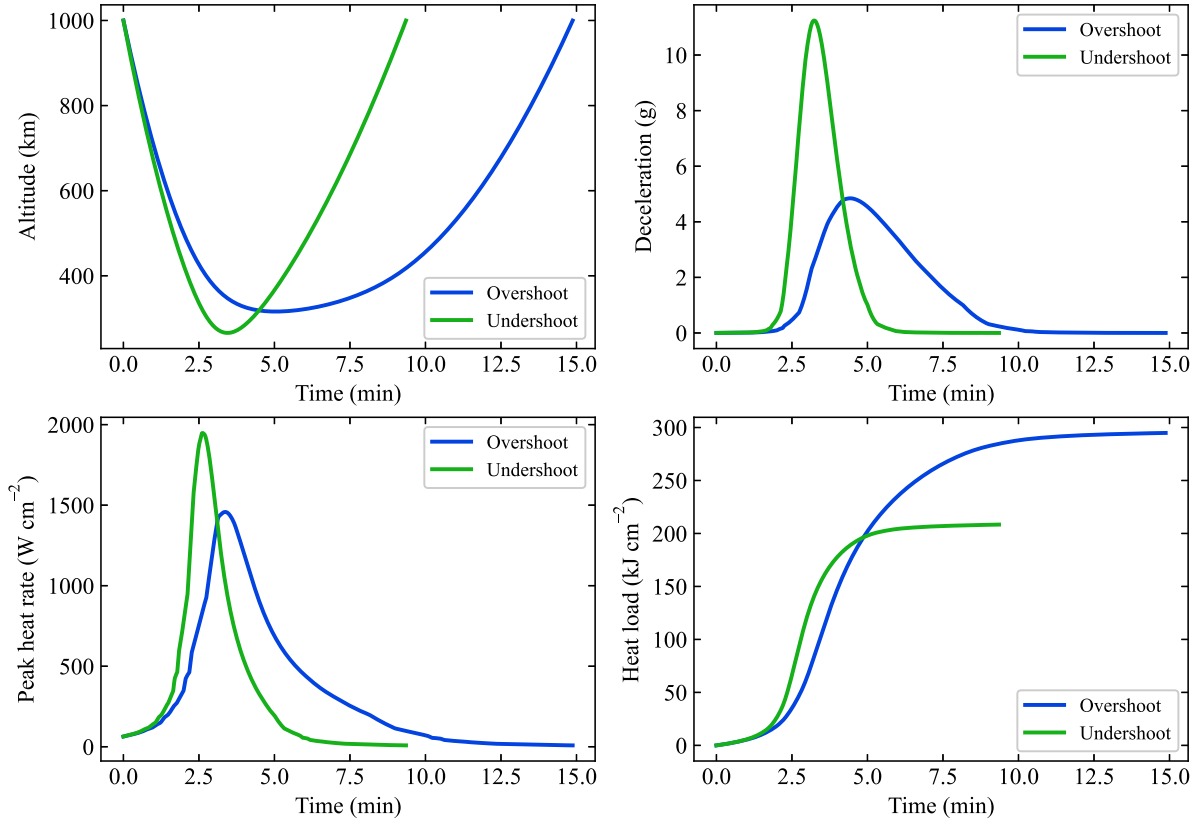
to avoid that the more or less energetic particles circling in the radiation belts around the giant planets (Schubert & Soderlund 2011; Masters, Ioannou & Rayns 2022) damage the instrumentation of the spacecraft, which would likely happen if the latter steadily moved at the same (small) distance from the planet in a circular, low orbit. Another drawback of an (almost) circular orbit is that the spacecraft would go behind Uranus every few hours, making, thus, difficult to communicate with the Earth. Another issue is that low orbits must avoid the Uranian ring system. Should the probe move completely inside the rings, the extended Uranian atmosphere would cause its orbit to quickly decay. Moreover, there would exist a significant risk of frequent ring plane crossing.

Here, it is shown how the technique of aerocapture (Cruz 1979; Leszczynski 1998; Spilker et al. 2019; Saikia et al. 2021; Girija et al. 2022) allows to insert a larger orbiter into an elliptical  $2000 \times 100\,000$  km orbit, and a small secondary satellite into a closer and less elliptical  $2000 \times 10\,000$  km orbit at Uranus; the orbital effects of interest for such orbital geometries were studied in Section 5. As shown by equation (57) and equation (58), the resulting orbital periods are relatively short, if compared to Juno or Cassini Grand Finale orbits. Such a feature may be challenging for spacecraft operations, and orbital maneuvers should be carefully planned.

The mission and aerocapture vehicle design presented in Girija (2023) to achieve a  $4000 \times 550\,000$  km trajectory is adapted here to achieve a smaller  $2000 \times 100\,000$  km elliptical, polar orbit. Its apocentre is just outside the Uranian ring system. Such a relatively small orbit is beyond the reach of conventional propulsive insertion due to the large  $\Delta V$  required, but it turns out to be feasible using aerocapture. Using a vehicle with lift-to-drag ( $L/D$ ) = 0.24, the theoretical corridor width (TCW) for aerocapture is  $1.22^\circ$ , which is sufficient to accommodate delivery errors and atmospheric uncertainties. Fig. 6 shows the overshoot and undershoot limit trajectories for an aerocapture vehicle targeting a  $100\,000$  km apocentre altitude at atmospheric exit using the aerocapture mission analysis tool by Girija et al. (2021).

The peak heat rate is estimated to be in the range of  $1500$  to  $2000 \text{ W cm}^{-2}$ , which is well within the capability of the Heatshield for Extreme Entry Environment Technology (Venkatapathy et al. 2020). At the first apocentre, the spacecraft performs a  $120 \text{ m s}^{-1}$  propulsive burn to raise the pericentre outside the atmosphere and achieve its  $2000 \times 100\,000$  km elliptical capture orbit.

Using a different aerocapture vehicle design known as drag modulation, it is possible to send a small orbiter with, say, a mass of  $200$  kg, into a  $2000 \times 10\,000$  km as a secondary spacecraft carried by the main orbiter. The orbit is so small that the  $\Delta V$  for its insertion exceeds  $12 \text{ km s}^{-1}$ . However, with aerocapture such an orbital configuration is achievable. It would also be considered too risky for a large orbiter, and presents engineering challenges such as the spacecraft frequently going behind Uranus as viewed from Earth and orbit decay due to drag from the upper atmosphere. A small spacecraft can accommodate the risks while providing measurements from the low-circular orbit which cannot be made otherwise. The small orbiter will detach from the mothership a few weeks prior to entry, and use aerocapture to insert into a



**Figure 6.** Overshoot and undershoot trajectories for a Mars Science Laboratory-derived aerocapture vehicle with lift-to-drag  $L/D = 0.24$ , ballistic coefficient  $\beta = 146 \text{ kg m}^{-2}$ , spacecraft nose radius  $R_n = 1.125 \text{ m}$ ,  $h_{\text{apo}} = 100\,000 \text{ km}$ . The independent variable on the horizontal axes is the time since entry interface, defined at 1000 km above the 1 bar pressure level, while the unit of measure for the deceleration denoted as ‘g’ in the vertical axis of the panel in the upper right corner is the mean Earth’s acceleration of gravity.

near-circular orbit around Uranus. The apocentre is just inside the rings, which start at  $\simeq 12\,500 \text{ km}$ . Using a drag modulation vehicle with<sup>8</sup>  $\beta_1 = 10 \text{ kg m}^{-2}$  and  $\beta_2/\beta_1 = 7.5$  offers a TCW of  $0.70^\circ$  for aerocapture to a  $h_{\text{apo}} = 10\,000 \text{ km}$ . Fig. 7 shows the nominal aerocapture trajectory of the latter. At the first pericentre, the spacecraft performs a  $200 \text{ m s}^{-1}$  propulsive burn to achieve its initial capture orbit of  $2000 \times 10\,000 \text{ km}$ .

## 7 SUMMARY AND CONCLUSIONS

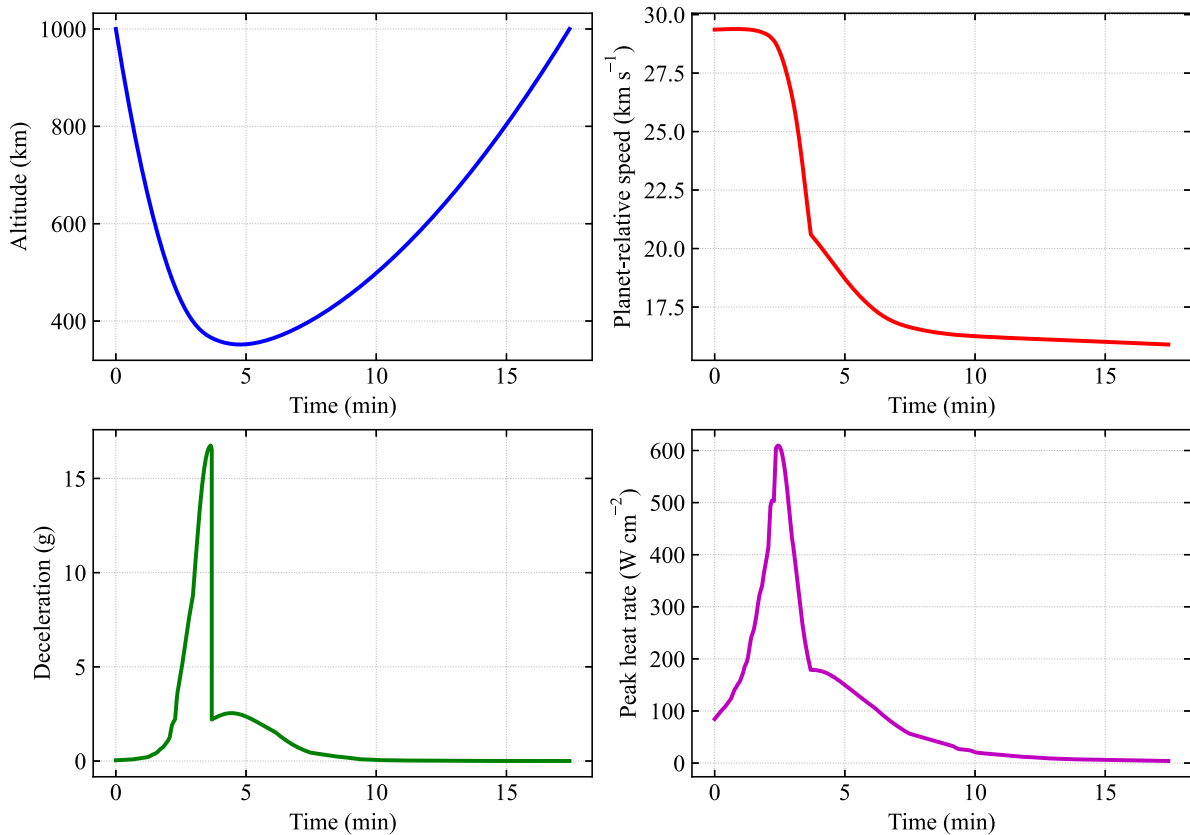
A mission concept, provisionally dubbed EURO, based on the use of a spacecraft around Uranus to measure the planet’s angular momentum  $S_{\delta}$  by means of its general relativistic LT orbital precessions independently of the Newtonian ones induced by the planetary gravity field’s zonal multipoles  $J_\ell$ ,  $\ell = 2, 3, 4, \dots$  was presented.

The orbital geometry allowing, in principle, the fulfilment of such a goal implies an orbital plane perpendicular to the CE and containing the Uranian spin axis  $\hat{k}_{\delta}$ . Moreover, its node  $\Omega$  is equal to the RA  $\alpha_{\delta}$  of  $\hat{k}_{\delta}$ . Indeed, in this case, the classical precessions of the orbital inclination  $I$  and of  $\Omega$  due to the even and odd zonals of the planet’s gravity field ideally vanish, contrary to the gravitomagnetic ones. The situation is reversed for the precessions of the argument of pericentre  $\omega$ .

For a  $2000 \times 100\,000 \text{ km}$  orbit, it turns out that the general relativistic rates of  $I$  and  $\Omega$  amount to  $\simeq 16\text{--}59 \text{ mas yr}^{-1}$ , while the size of the Newtonian pericentre precession due to  $J_2$  is as large as  $\simeq 173^\circ \text{ yr}^{-1}$ . An approximate evaluation of the LT net shift per orbit of the range-rate yields  $\simeq 2 \times 10^{-4} \text{ mm s}^{-1}$  for a suitable choice of the initial conditions, with a peak-to-peak amplitude up to  $\simeq 1.5 \times 10^{-3} \text{ mm s}^{-1}$  for a single passage at the pericentre few hours long. Such figures are demanding with respect to the  $\simeq 10^{-2} \text{ mm s}^{-1}$  accuracy level of, say, Juno, posing a challenge to the success of the mission. By lowering the apocentre height down to  $10\,000 \text{ km}$ , it is possible to increase the relativistic signatures to the level of  $\simeq 100\text{--}400 \text{ mas yr}^{-1}$ .

A major source of systematic bias is represented by the accuracies with which the orientation of  $\hat{k}_{\delta}$  and  $I$  should be determined. It turns out that they are of the order of  $\simeq 0.1\text{--}1$  and  $\simeq 1\text{--}10 \text{ mas}$ , respectively; the pole of Uranus is currently known with an accuracy of  $\simeq 7 \text{ arcsec}$

<sup>8</sup>The parameters  $\beta_1$  and  $\beta_2$  are the values of the ballistic coefficient  $\beta$  before and after the drag skirt separation, respectively, for drag modulation aerocapture. The ballistic coefficient ratio  $\beta_2/\beta_1$  is a key design parameter indicating how much control authority the vehicle has.



**Figure 7.** Nominal drag modulation trajectory for a vehicle with ballistic coefficient before the drag skirt separation  $\beta_1 = 10 \text{ kg m}^{-2}$ , ballistic coefficient ratio  $\beta_2/\beta_1 = 7.5$ , and apocentre altitude  $h_{\text{apo}} = 10000 \text{ km}$ . The independent variable on the horizontal axes is the time since entry interface, defined at 1000 km above the 1 bar pressure level, while the unit of measure for the deceleration denoted as ‘g’ in the vertical axis of the panel in the lower left corner is the mean Earth’s acceleration of gravity.

due to the lack of dedicated, *in situ* mission(s). To this aim, it is conceivable that a possible ‘phase 1’ of the EURO mission may allow for good enough measurements of  $\hat{k}_\delta$  and  $I$  by standard geodetic techniques.

Using the LT effect to dynamically measure the spin angular momentum of Uranus is definitely a challenging task.

## ACKNOWLEDGEMENTS

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## DATA AVAILABILITY

Aerocapture trajectory results presented in the study can be reproduced using Jupiter Notebooks available at <https://amat.readthedocs.io/en/master/other-notebooks.html#euro-uranus-orbiter>. The study results were made with AMAT v2.2.22. An archived version of AMAT v2.2.22 is available at Zenodo: DOI:10.5281/zenodo.7542714.

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## APPENDIX A: THE AVERAGED RATES OF $I$ , $\Omega$ , and $\omega$ DUE TO SOME EVEN AND ODD ZONAL HARMONICS

The averaged rates of the inclination  $I$ , of the longitude of the ascending node  $\Omega$ , and of the argument of pericentre  $\omega$  induced by the zonal harmonics  $J_\ell$  of degree  $\ell = 2, 3, 4, \dots$  of the multipolar expansion of the classical part of the gravitational potential of a non-spherical body can be computed, e.g. with the planetary equations in the form of Lagrange (Bertotti et al. 2003a):

$$j^{J_\ell} = \frac{1}{n_b a^2 \sqrt{1-e^2} \sin I} \left( \frac{\partial \langle \Delta U_\ell \rangle}{\partial \Omega} - \cos I \frac{\partial \langle \Delta U_\ell \rangle}{\partial \omega} \right), \quad (\text{A1})$$

$$\dot{\Omega}^{J_\ell} = - \frac{1}{n_b a^2 \sqrt{1-e^2} \sin I} \frac{\partial \langle \Delta U_\ell \rangle}{\partial I}, \quad (\text{A2})$$

$$\dot{\omega}^{J_\ell} = \frac{\cot I}{n_b a^2 \sqrt{1-e^2}} \frac{\partial \langle \Delta U_\ell \rangle}{\partial I} - \frac{\sqrt{1-e^2}}{n_b a^2 e} \frac{\partial \langle \Delta U_\ell \rangle}{\partial e}, \quad (\text{A3})$$

where

$$\langle U_\ell \rangle = \frac{n_b}{2\pi} \int_{f_0}^{f_0+2\pi} \Delta U_\ell \frac{dt}{df} df. \quad (\text{A4})$$

In equation (A4), the instantaneous correction  $\Delta U_\ell$  of degree  $\ell$  of the Newtonian monopole, to be evaluated on to an unperturbed Keplerian ellipse by means of

$$r = \frac{p}{1+e \cos f}, \quad (\text{A5})$$

$$\frac{dt}{df} = \frac{(1-e^2)^{3/2}}{n_b (1+e \cos f)^2}, \quad (\text{A6})$$

is (Bertotti et al. 2003a)

$$\Delta U_\ell = \frac{\mu}{r} J_\ell \left( \frac{R}{r} \right)^\ell \mathcal{P}_\ell(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}). \quad (\text{A7})$$

In equation (A7),  $\mathcal{P}_\ell(\dots)$  is the Legendre polynomial of degree  $\ell$ , and its argument, i.e. the cosine of the angle between the primary's spin axis  $\hat{\mathbf{k}}$  and the position vector  $\mathbf{r}$  of the test particle, can be written as (Soffel 1989; Brumberg 1991; Soffel & Han 2019)

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{r}} = (\hat{\mathbf{k}} \cdot \hat{\mathbf{l}}) \cos u + (\hat{\mathbf{k}} \cdot \hat{\mathbf{m}}) \sin u, \quad (\text{A8})$$

where

$$u \doteq \omega + f \quad (\text{A9})$$

is the argument of latitude.

For  $\ell = 2, 3, 4, \dots, 8$ , it turns out that the averaged rates of  $I$  and  $\Omega$ , computed with equations (46)–(47), vanish, contrary to the pericentre whose precessions are

$$\dot{\omega}^{J_2} = -\frac{3}{4} n_b J_2 \left( \frac{R}{p} \right)^2, \quad (\text{A10})$$

$$e \dot{\omega}^{J_3} = -\frac{3}{8} n_b J_3 \left( \frac{R}{p} \right)^3 (1 + 4e^2) \cos \xi, \quad (\text{A11})$$

$$\dot{\omega}^{J_4} = -\frac{15}{128} n_b J_4 \left( \frac{R}{p} \right)^4 [12 + 9e^2 + 2(2 + 5e^2) \cos 2\xi], \quad (\text{A12})$$

$$e \dot{\omega}^{J_5} = -\frac{15}{256} n_b J_5 \left( \frac{R}{p} \right)^5 \cos \xi [8 + 75e^2 + 22e^4 + 14(e^2 + 2e^4) \cos 2\xi], \quad (\text{A13})$$

$$\dot{\omega}^{J_6} = -\frac{105}{4096} n_b J_6 \left( \frac{R}{p} \right)^6 [80 + 50e^2(4 + e^2) + 10(4 + 22e^2 + 7e^4) \cos 2\xi + 3e^2(4 + 7e^2) \cos 4\xi], \quad (\text{A14})$$

$$e \dot{\omega}^{J_7} = -\frac{21}{8192} n_b J_7 \left( \frac{R}{p} \right)^7 \cos \xi [200 + 3360e^2 + 3570e^4 + 592e^6 + 6e^2(180 + 685e^2 + 136e^4) \cos 2\xi + 33e^4(5 + 8e^2) \cos 4\xi], \quad (\text{A15})$$

$$\begin{aligned} \dot{\omega}^{J_8} = & \frac{63}{262144} n_b J_8 \left( \frac{R}{p} \right)^8 (-175 \{64 + 7e^2 [48 + 5e^2(8 + e^2)]\} - \\ & -70(96 + 944e^2 + 970e^4 + 135e^6) \cos 2\xi - 154e^2(40 + 128e^2 + 27e^4) \cos 4\xi - \\ & -286e^4(2 + 3e^2) \cos 6\xi), \end{aligned} \quad (\text{A16})$$

where

$$\xi \doteq \delta - \omega. \quad (\text{A17})$$

Note that the precessions induced by the odd zonals, i.e. equation (A11), equation (A13), and equation (A15), vanish if calculated with equation (96).

## APPENDIX B: THE RANGE-RATE PERTURBATION

The velocity vector  $\mathbf{v}$  of a test particle moving along an unperturbed Keplerian ellipse around a massive primary is

$$\mathbf{v} = \{v_x, v_y, v_z\}, \quad (\text{B1})$$

with (Kopeikin et al. 2011; Poisson & Will 2014)

$$v_x = -\frac{n_b a}{\sqrt{1-e^2}} [\cos \Omega (e \sin \omega + \sin u) + \cos I \sin \Omega (e \cos \omega + \cos u)], \quad (\text{B2})$$

$$v_y = -\frac{n_b a}{\sqrt{1-e^2}} [-\cos I \cos \Omega (e \cos \omega + \cos u) + \sin \Omega (e \sin \omega + \sin u)], \quad (\text{B3})$$

$$v_z = \frac{n_b a}{\sqrt{1-e^2}} \sin I (e \cos \omega + \cos u). \quad (\text{B4})$$

The instantaneous change  $\Delta \mathbf{v}$  of  $\mathbf{v}$  has, then, to be computed as<sup>9</sup>

$$\Delta \mathbf{v} = \left\{ \sum_{\kappa} \frac{\partial v_x}{\partial \kappa} \Delta \kappa (f), \sum_{\kappa} \frac{\partial v_y}{\partial \kappa} \Delta \kappa (f), \sum_{\kappa} \frac{\partial v_z}{\partial \kappa} \Delta \kappa (f) \right\}, \quad \kappa = a, e, I, \Omega, \omega, f. \quad (\text{B5})$$

In equation (B5), the instantaneous shift  $\Delta \kappa (f)$  of any perturbed osculating orbital element  $\kappa$  among  $a, e, I, \Omega, \omega$  can be analytically worked out as

$$\Delta \kappa (f) = \int_{f_0}^f \frac{d\kappa}{dt} \frac{dt}{df'} df', \quad (\text{B6})$$

where  $d\kappa/dt$  is the right-hand side of the equation for the osculating element  $\kappa$  in the Euler–Gauss form (Soffel 1989; Brumberg 1991; Kopeikin et al. 2011; Soffel & Han 2019), calculated with the disturbing acceleration  $\mathbf{A}$  at hand and evaluated on to the unperturbed Keplerian ellipse characterized by equations (A5)–(A6). As per the variation  $\Delta f$  of the true anomaly, it is usually expressed in terms of the changes of the mean anomaly  $\mathcal{M}$  and of the eccentricity  $e$  as follows. First, the true anomaly  $f$  is written as a function of the eccentric anomaly  $E$  as (Capderou 2005)

$$\tan \left( \frac{f}{2} \right) = \sqrt{\frac{1+e}{1-e}} \tan \left( \frac{E}{2} \right). \quad (\text{B7})$$

Then, the shift of  $f$  is calculated by differentiating equation (B7) with respect to  $e$  and  $E$  as

$$\Delta f = \frac{\partial f}{\partial e} \Delta e + \frac{\partial f}{\partial E} \Delta E = \frac{\sin E}{\sqrt{1-e^2} (1-e \cos E)} \Delta e + \sqrt{\frac{1+e}{1-e}} \frac{1-e}{(1-e \cos E)} \Delta E. \quad (\text{B8})$$

In turn, from the Kepler equation (Capderou 2005)

$$\mathcal{M} = E - e \sin E, \quad (\text{B9})$$

one gets

$$\Delta E = \frac{\Delta \mathcal{M} + \sin E \Delta e}{1 - e \cos E}. \quad (\text{B10})$$

Finally, by inserting equation (B10) in equation (B8) and using (Capderou 2005)

$$r = a (1 - e \cos E), \quad (\text{B11})$$

$$\sin E = \frac{\sqrt{1-e^2} \sin f}{1+e \cos f}, \quad (\text{B12})$$

$$\cos E = \frac{e + \cos f}{1+e \cos f}, \quad (\text{B13})$$

one obtains

$$\Delta f = \left( \frac{a}{r} \right) \left[ \sin f \left( 1 + \frac{r}{p} \right) \Delta e + \sqrt{1-e^2} \left( \frac{a}{r} \right) \right] \Delta \mathcal{M}, \quad (\text{B14})$$

in agreement with Casotto (1993, equation A.6).

The instantaneous perturbations  $\Delta v_R (f)$ ,  $\Delta v_T (f)$ , and  $\Delta v_N (f)$  of the radial ( $R$ ), transverse ( $T$ ), and out-of-plane ( $N$ ) components of the velocity are worked out by projecting equation (B5) on to the radial, transverse, and normal directions given by the unit vectors (Soffel 1989; Brumberg 1991; Soffel & Han 2019):

$$\mathbf{u}_R = \{ \cos \Omega \cos u - \cos I \sin \Omega \sin u, \sin \Omega \cos u + \cos I \cos \Omega \sin u, \sin I \sin u \}, \quad (\text{B15})$$

<sup>9</sup>In calculating the partial derivatives with respect to  $a, n_b$  has to be considered as a function of  $a$ .

$$\mathbf{u}_T = \{-\cos \Omega \sin u - \cos I \sin \Omega \cos u, -\sin \Omega \sin u + \cos I \cos \Omega \cos u, \sin I \cos u\}, \quad (\text{B16})$$

$$\mathbf{u}_N = \{\sin I \sin \Omega, -\sin I \cos \Omega, \cos I\}. \quad (\text{B17})$$

As a result, one gets

$$\begin{aligned} \Delta v_R(f) = \Delta \mathbf{v} \cdot \mathbf{u}_R = & -\frac{n_b \sin f}{\sqrt{1-e^2}} \left[ \frac{e \Delta a(f)}{2} + \frac{a^2 \Delta e(f)}{r} \right] - \\ & -\frac{n_b a^2 \sqrt{1-e^2}}{r} [\cos I \Delta \Omega(f) + \Delta \omega(f)] - \frac{n_b a^3}{r^2} \Delta \mathcal{M}(f), \end{aligned} \quad (\text{B18})$$

$$\begin{aligned} \Delta v_T(f) = \Delta \mathbf{v} \cdot \mathbf{u}_T = & -\frac{n_b a \sqrt{1-e^2}}{2r} \Delta a(f) + \frac{n_b a (e + \cos f)}{(1-e^2)^{3/2}} \Delta e(f) + \\ & + \frac{n_b a e \sin f}{\sqrt{1-e^2}} [\cos I \Delta \Omega(f) + \Delta \omega(f)], \end{aligned} \quad (\text{B19})$$

$$\Delta v_N(f) = \Delta \mathbf{v} \cdot \mathbf{u}_N = \frac{n_b a}{\sqrt{1-e^2}} [(\cos u + e \cos \omega) \Delta I(f) + \sin I (\sin u + e \sin \omega) \Delta \Omega(f)], \quad (\text{B20})$$

which agree just with Casotto (1993, equations 33–35). Thus, the perturbation of the velocity vector can be expressed as

$$\Delta \mathbf{v} = \Delta v_R \mathbf{u}_R + \Delta v_T \mathbf{u}_T + \Delta v_N \mathbf{u}_N, \quad (\text{B21})$$

where  $\Delta v_R$ ,  $\Delta v_T$ , and  $\Delta v_N$  are given by equations (B18)–(B20), and the unit vectors  $\mathbf{u}_R$ ,  $\mathbf{u}_T$ , and  $\mathbf{u}_N$  are as per equations (B15)–(B17).

In case of a spacecraft orbiting a distant planet  $P$ , the dynamical<sup>10</sup> component of the range-rate  $\dot{\rho}$  is the projection of the planetocentric velocity  $\mathbf{v}$  on to the unit vector  $\hat{\boldsymbol{\rho}}$  of the line of sight which can be approximated with the opposite of the versor of the geocentric position vector  $\hat{\mathbf{R}}$  of  $P$ . Usually, the latter remains constant during a fast passage of the probe at the pericentre, or even during a full orbital revolution. In terms of the RA  $\phi$  and DEC  $\chi$  of  $P$ , it can be written as

$$\hat{\boldsymbol{\rho}} \simeq -\hat{\mathbf{R}} = -\{\cos \chi \cos \phi, \cos \chi \sin \phi, \sin \chi\}. \quad (\text{B22})$$

Thus, the range-rate perturbation  $\Delta \dot{\rho}$  can be analytically calculated by means of equation (B21) and equation (B22) as

$$\Delta \dot{\rho} = \Delta \mathbf{v} \cdot \hat{\boldsymbol{\rho}} \simeq -\Delta \mathbf{v} \cdot \hat{\mathbf{R}}. \quad (\text{B23})$$

## APPENDIX C: NAMING THE APSIDAL POSITIONS IN THE CASE OF URANUS

Here, some proposals for properly naming the pericentre and apocentre in the case of Uranus are given.

The first suggestion is ‘periouranon’ and ‘apouranon’, from  $\pi\epsilon\rho\acute{\iota} + \text{O}\ddot{\upsilon}\rho\alpha\nu\acute{o}\varsigma$  ( $\pi\epsilon\rho\acute{\iota} + \bar{\Upsilon}\alpha\nu\acute{o}\varsigma$ ) and  $\acute{\alpha}\pi\acute{o} + \text{O}\ddot{\upsilon}\rho\alpha\nu\acute{o}\varsigma$  ( $\acute{\alpha}\pi\acute{o} + \bar{\Upsilon}\alpha\nu\acute{o}\varsigma$ ), respectively. Indeed,  $\text{O}\ddot{\upsilon}\rho\alpha\nu\acute{o}\varsigma$  is the Greek god personifying the sky after whom the seventh planet of the Solar system was named, while  $\pi\epsilon\rho\acute{\iota}$  (+ accusative) and  $\acute{\alpha}\pi\acute{o}$  (+ genitive) are prepositions meaning *around*, *near*, *about*, *from*, and *away from*, respectively. One might be tempted to propose the term ‘aphouranon’, as in ‘aphelion’ for the Sun. Actually, the former should be avoided since the apocopic form  $\acute{\alpha}\phi$  ( $\acute{\alpha}\phi$ ) of  $\acute{\alpha}\pi\acute{o}$  is used before a vowel with rough breathing (‘), as just in ‘ $\text{H}\eta\lambda\acute{\iota}\omicron\varsigma$  (Hélios), while  $\text{O}\ddot{\upsilon}\rho\alpha\nu\acute{o}\varsigma$  has the smooth breathing (‘).

As an alternative proposal, ‘pericælum’ and ‘apocælum’ could also be considered since *Cælūs* is the Roman counterpart of  $\text{O}\ddot{\upsilon}\rho\alpha\nu\acute{o}\varsigma$ .

Should one rely upon the colour of the planet, ‘pericærulum’ and ‘apocærulum’ could be adopted, from the Latin adjective *cærūlus* meaning, among other things, just ‘greenish-blue’.

This paper has been typeset from a  $\text{T}_{\text{E}}\text{X}/\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  file prepared by the author.

<sup>10</sup>It means that it is solely due to the planetocentric orbital motion of the probe; it neglects all the special and general relativistic effects connected with the propagation of the electromagnetic waves through a variable gravitational field (Kopeikin et al. 2011).